

## ラプラス演算子の極座標表現

$$\begin{aligned}\nabla &\equiv \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \\ &= (\nabla x) \frac{\partial}{\partial x} + (\nabla y) \frac{\partial}{\partial y} + (\nabla z) \frac{\partial}{\partial z}\end{aligned}\quad (1)$$

$$\nabla x = \left( \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) x$$

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases} \quad (2)$$

ここで

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} + \frac{\partial z}{\partial r} \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial \varphi} = \frac{\partial x}{\partial \varphi} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi} \frac{\partial}{\partial y} + \frac{\partial z}{\partial \varphi} \frac{\partial}{\partial z}$$

まとめて行列表現すると

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$= \begin{pmatrix} \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \\ r \cos \theta \sin \varphi & r \cos \theta \sin \varphi & -r \sin \theta \\ r \sin \theta \sin \varphi & r \sin \theta \cos \varphi & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \varphi & \frac{1}{r} \sin \theta \sin \varphi & -\frac{\sin \varphi}{r \sin \theta} \\ \sin \theta \sin \varphi & \frac{1}{r} \cos \theta \sin \varphi & \frac{\cos \varphi}{r \sin \theta} \\ \cos \theta & -\frac{1}{r} \sin \theta \cos \varphi & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \varphi} \end{pmatrix}$$

これはまた

$$\begin{aligned}
\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} &= \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} & \frac{\partial \varphi}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} & \frac{\partial \varphi}{\partial y} \\ \frac{\partial r}{\partial z} & \frac{\partial \theta}{\partial z} & \frac{\partial \varphi}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \varphi} \end{pmatrix} \\
\therefore \nabla &= \hat{x} \left( \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x} \frac{\partial}{\partial \varphi} \right) \\
&\quad + \hat{y} \left( \frac{\partial r}{\partial y} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial y} \frac{\partial}{\partial \varphi} \right) \\
&\quad + \hat{z} \left( \frac{\partial r}{\partial z} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z} \frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial z} \frac{\partial}{\partial \varphi} \right) \\
&= \left( \hat{x} \frac{\partial r}{\partial x} + \hat{y} \frac{\partial r}{\partial y} + \hat{z} \frac{\partial r}{\partial z} \right) \frac{\partial}{\partial r} \\
&\quad + \left( \hat{x} \frac{\partial \theta}{\partial x} + \hat{y} \frac{\partial \theta}{\partial y} + \hat{z} \frac{\partial \theta}{\partial z} \right) \frac{\partial}{\partial \theta} \\
&\quad + \left( \hat{x} \frac{\partial \varphi}{\partial x} + \hat{y} \frac{\partial \varphi}{\partial y} + \hat{z} \frac{\partial \varphi}{\partial z} \right) \frac{\partial}{\partial \varphi} \\
&= (\nabla r) \frac{\partial}{\partial r} + (\nabla \theta) \frac{\partial}{\partial \theta} + (\nabla \varphi) \frac{\partial}{\partial \varphi} \\
\nabla r &= \hat{x} \sin \theta \cos \varphi + \hat{y} \sin \theta \sin \varphi + \hat{z} \cos \theta \\
|\nabla r| &= 1 \quad \therefore \nabla r = \hat{r} \\
\nabla \theta &= \hat{x} \frac{1}{r} \cos \theta \cos \varphi + \hat{y} \frac{1}{r} \cos \theta \sin \varphi - \hat{z} \frac{\sin \theta}{r} \\
|\nabla \theta| &= \frac{1}{r} \quad \therefore \hat{\theta} = r(\nabla \theta) \\
\nabla \varphi &= -\hat{x} \frac{\sin \varphi}{r \sin \theta} + \hat{y} \frac{\cos \varphi}{r \sin \theta} \\
|\nabla \varphi| &= \frac{1}{r \sin \theta} \quad \therefore r \sin \theta (\nabla \varphi) \\
\nabla &= (\nabla r) \frac{\partial}{\partial r} + (\nabla \theta) \frac{\partial}{\partial \theta} + (\nabla \varphi) \frac{\partial}{\partial \varphi} \\
&= \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\varphi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}
\end{aligned}$$

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \\ \cos \theta \cos \varphi & \cos \theta \sin \varphi & -\sin \theta \\ -\sin \varphi & \cos \varphi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}$$

$$\begin{aligned} \frac{\partial \hat{\mathbf{r}}}{\partial r} &= 0 \\ \frac{\partial \hat{\mathbf{r}}}{\partial \theta} &= \hat{\mathbf{x}} \cos \theta \cos \varphi + \hat{\mathbf{y}} \cos \theta \sin \varphi - \hat{\mathbf{z}} \sin \theta = \hat{\boldsymbol{\theta}} \\ \frac{\partial \hat{\mathbf{r}}}{\partial \varphi} &= -\hat{\mathbf{x}} \sin \theta \sin \varphi + \hat{\mathbf{y}} \sin \theta \cos \varphi = \sin \theta \hat{\boldsymbol{\varphi}} \\ \frac{\partial \hat{\boldsymbol{\theta}}}{\partial r} &= 0 \\ \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} &= -\hat{\mathbf{x}} \sin \theta \cos \varphi - \hat{\mathbf{y}} \sin \theta \sin \varphi - \hat{\mathbf{z}} \cos \theta = -\hat{\mathbf{r}} \\ \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \varphi} &= -\hat{\mathbf{x}} \cos \theta \sin \varphi + \hat{\mathbf{y}} \cos \theta \cos \varphi = \cos \theta \hat{\boldsymbol{\varphi}} \\ \frac{\partial \hat{\boldsymbol{\varphi}}}{\partial r} &= 0, \quad \frac{\partial \hat{\boldsymbol{\varphi}}}{\partial \theta} = 0 \\ \frac{\partial \hat{\boldsymbol{\varphi}}}{\partial \varphi} &= -\hat{\mathbf{x}} \cos \varphi + \hat{\mathbf{y}} \sin \varphi = -\sin \theta \hat{\mathbf{r}} - \cos \theta \hat{\boldsymbol{\theta}} \\ \Delta = \nabla \cdot \nabla &= \left( \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}} \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right) \cdot \left( \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}} \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right) \\ &= \hat{\mathbf{r}} \frac{\partial}{\partial r} \cdot \left( \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}} \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right) \\ &\quad + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} \cdot \left( \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}} \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right) \\ &\quad + \hat{\boldsymbol{\varphi}} \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \cdot \left( \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}} \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right) \\ &= \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \frac{\partial^2}{\partial r^2} + \hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial \theta} \right) + \hat{\mathbf{r}} \cdot \hat{\boldsymbol{\varphi}} \frac{\partial}{\partial r} \left( \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right) \\ &\quad + \hat{\boldsymbol{\theta}} \frac{1}{r} \cdot \left( \hat{\boldsymbol{\theta}} \frac{\partial}{\partial r} + \hat{\mathbf{r}} \frac{\partial^2}{\partial \theta \partial r} \right) + \hat{\boldsymbol{\theta}} \frac{1}{r} \left( -\hat{\mathbf{r}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial^2}{\partial \theta^2} \right) \\ &\quad + \hat{\boldsymbol{\theta}} \frac{1}{r} \cdot \left( \hat{\boldsymbol{\varphi}} \frac{\partial}{\partial \theta} \left( \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right) \right) \\ &\quad + \hat{\boldsymbol{\varphi}} \frac{1}{r \sin \theta} \left( \sin \theta \hat{\boldsymbol{\varphi}} \frac{\partial}{\partial r} \right) + \hat{\boldsymbol{\varphi}} \frac{1}{r \sin \theta} \cdot \left( \cos \theta \hat{\boldsymbol{\varphi}} \frac{1}{r} \frac{\partial}{\partial \theta} \right) + \hat{\boldsymbol{\varphi}} \frac{1}{r \sin \theta} \cdot \left( \hat{\boldsymbol{\varphi}} \frac{1}{r \sin \theta} \frac{\partial^2}{\partial \varphi^2} \right) \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \end{aligned}$$